Lieb-Robinson Bounds for Harmonic and Anharmonic Lattice Systems

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February 2, 2008

Abstract

We prove Lieb-Robinson bounds for systems defined on infinite dimensional Hilbert spaces and described by unbounded Hamiltonians. In particular, we consider harmonic and certain anharmonic lattice systems.

1 Introduction

An important class of systems in statistical mechanics is described by the (an)harmonic lattice Hamiltonians, which have a continuous degree of freedom, thought of as a particle trapped in a potential, at each site of a lattice. The particles interact by a linear or non-linear force. For example, such models are thought to describe the emergence of macroscopic non-equilibrium phenomena, such as heat conduction, from many-body Hamiltonian dynamics [24, 2], the understanding of which is one of the long-standing open problems in mathematical statistical mechanics [3].

In terms of technical difficulty, lattice oscillator models are intermediate between spin systems, where the degrees of freedom, each described by a finite-dimensional Hilbert space, are labeled by a discrete set, usually a lattice such as $\mathbb{Z}^\nu$, on the one hand, and particles in continuous space, which necessarily have an infinite-dimensional state space, on the other hand. Even in the classical case lattice oscillator systems are significantly more difficult to study than spin systems, and also for them more is known than for particle models in the continuum. E.g., the existence of the dynamics in the thermodynamics limit was studied by Lanford, Lebowitz, and Lieb in [15].

In this paper we focus on an essential locality property of the dynamics of quantum harmonic and anharmonic lattice models. Since these are non-relativistic models there is no *a priori* bound on the speed of propagation of signals in these systems. In the case of quantum spin systems with finite-range interactions, Lieb and Robinson [16] showed that there is nevertheless an upper bound on the speed of propagation in the sense that disturbances in the system remain confined in a "light" cone up to small corrections that decay at least exponentially fast away from the light cone. This is the so-called Lieb-Robinson bound which is an upper bound on the speed of propagation.

In the past few years several generalizations, improvements, and applications of Lieb-Robinson type bounds have appeared. This work can be regarded as one further extension, going for the first time beyond the realm of quantum spin systems. Here, by quantum spin system we mean any quantum system with a finite dimensional Hilbert space of states. For example, a quantum spin system over a finite subset $\Lambda \subset \mathbb{Z}^\nu$ is described on the Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x \qquad \text{with } \mathcal{H}_x = \mathbb{C}^{n_x}$$

where the dimensions $2 \leq n_x < \infty$ are related to the magnitude of the spin at site $x \in \Lambda$. The algebra of observables for this quantum spin system is then given by

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda)$$

where $\mathcal{B}(\mathcal{H}_x)$ is the space of bounded operators on $\mathcal{H}_x$ (that is the space of all $n_x \times n_x$ matrices). The Hamiltonian of the quantum spin system is usually written in the form

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$$

where the interaction $\Phi : 2^\Lambda \to \mathcal{A}_\Lambda$ is such that $\Phi(X)^* = \Phi(X) \in \mathcal{A}_X = \otimes_{x\in X}\mathcal{B}(\mathcal{H}_x)$ for all $X \subset \Lambda$. The time evolution associated with the Hamiltonian H_Λ is then the one-parameter group of automorphisms $\{\tau_t^\Lambda\}_{t\in\mathbb{R}}$ defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \qquad \text{for all } A \in \mathcal{A}_\Lambda\,.$$

For such systems, under appropriate conditions on the interactions $\Phi(X)$ (short-range conditions) it was first proved by Lieb and Robinson in [16], that, given $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$,

$$\|[\tau_t^\Lambda(A), B]\| \leq C\|A\|\|B\|\, e^{-\mu(d(X,Y)-v|t|)} \tag{1.1}$$

where $d(X,Y) = \min_{x \in X, y \in Y} |x-y|$ and $|x| = \sum_{j=1}^{\nu} |x_j|$. The physical interpretation of this inequality is straightforward; if two observables A and B are supported in disjoint regions, then even after evolving the observable A, apart from exponentially small contributions, their supports remain essentially disjoint up to times $t \leq d(X,Y)/v$. In other words, this bound asserts that the speed of propagation of perturbations in quantum spin systems is bounded.

In the original proof of the Lieb-Robinson bounds (see [16]), the constant C and the velocity v on the right hand side of (1.1) depended in a crucial way on $N = \max_{x \in \Lambda} n_x$, the maximal dimension of the different spin spaces. More recently, new Lieb-Robinson bounds of the form (1.1) were derived with a constant C and a velocity of propagation v independent of the dimension of the various spin spaces [14, 19]. This new version of the Lieb-Robinson bounds allowed for new applications, such as, among other results, a proof of the Lieb-Schutz-Mattis theorem in higher dimension, see [12, 20].

It seems natural to ask whether Lieb-Robinson bounds such as (1.1) can be extended to systems defined on infinite dimensional Hilbert spaces, and described by unbounded Hamiltonians. Although the constant C and the velocity v in (1.1) are independent of the dimension of the spin spaces, they depend on the operator norm of the interactions $\Phi(X)$; for this reason, if one deals with unbounded Hamiltonians, the methods developed in [19, 18, 14] cannot be applied directly. Nevertheless, in the present paper we prove that Lieb-Robinson bounds can be established for three different types of models with unbounded Hamiltonians, which we now present. For the precise statements see Sections 2, 3, and 4.

First, in Section 2, we consider systems defined on an infinite dimensional Hilbert space by Hamilton operators with possibly unbounded on-site terms but bounded interactions between sites. In this case, we show that the analysis of [19] goes through with only minor changes, and that Lieb-Robinson bounds can be proven in quite a large generality (see Theorem 2.1). A class of interesting examples of this are lattice oscillators coupled by bounded interactions. For a finite subset $\Lambda \subset \mathbb{Z}^{\nu}$, one considers the system defined on the Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} L^2(\mathbb{R}, \mathrm{d}q_x)$ by the Hamiltonian

$$H = \sum_{x \in \Lambda} p_x^2 + V(q_x) + \sum_{x,y \in \Lambda,\, |x-y|=1} \phi(q_x - q_y)$$

where $p_x = -i\, \mathrm{d}/\mathrm{d}q_x$, the real function V is such that $-\Delta_q + V(q)$ is a self-adjoint operator, and $\phi \in L^\infty(\mathbb{R})$ is real valued. Another commonly studied model that satisfies the conditions of this result is the so-called quantum rotor Hamiltonian of the form

$$H = -\sum_x \frac{\partial^2}{\partial \theta_x^2} + \sum_{x,y} J_{xy} \cos(\theta_x - \theta_y + \phi)$$

where θ_x is the angle associated with the rotor at site x, and J_{xy} are coupling constants assumed to vanish whenever $|x-y|$ exceeds a finite range R. Quantum rotor Hamiltonians are used to study a variety of physical situations such as Josephson junction arrays [1], the Bose-Hubbard model [22], and crystals consisting of molecules with rotor degrees of freedom [11].

Second, in Section 3, we consider harmonic lattice systems for which the Hamiltonian describes a system of linearly coupled harmonic oscillators situated at the points of a finite

subset $\Lambda \subset \mathbb{Z}^\nu$. The standard Hamiltonian is of the form

$$H^h = \sum_x p_x^2 + \omega^2 q_x^2 + \sum_{|x-y|=1} \sum_{j=1}^{\nu} \lambda_j (q_x - q_y)^2,$$

defined on a finite hypercube in $\mathbb{Z}^\nu$, with periodic boundary conditions. In this case, not only the on-site terms but also the interactions between sites are given by unbounded operators, and the analysis of [19] cannot be applied. As is well-known, the time evolution for harmonic systems can be computed explicitly (see Lemma 3.4), and the derivation of Lieb-Robinson bounds (in the form given in Theorem 3.1) reduces to the study of the asymptotic properties of certain Fourier sums (see Lemma 3.5).

Finally, in Section 4, we consider local anharmonic perturbations of the harmonic lattice system of the form

$$H = \sum_x p_x^2 + \omega^2 q_x^2 + \sum_{|x-y|=1} \sum_{j=1}^{\nu} \lambda_j (q_x - q_y)^2 + \sum_x V(q_x)\,.$$

Assuming that the local perturbation V is sufficiently weak (in an appropriate sense), and making use of an interpolation argument between the harmonic and the anharmonic time-evolution, we derive Lieb-Robinson bounds in Theorem 4.1.

Next, we discuss the classes of observables for which we obtain the Lieb-Robinson bounds in each of the three types of models. In the case of quantum spin systems, i.e., the case where the Hilbert spaces associated with a lattice site are all finite-dimensional, one proves Lieb-Robinson bounds for a pair of arbitrary observables A and B with finite supports (see (1.1)). It is not clear in general that such a result should be expected when the Hilbert spaces are infinite-dimensional and the Hamiltonians unbounded. If the unboundedness in the Hamiltonian is restricted to on-site terms while interactions between sites are bounded and of sufficiently short range, the standard Lieb-Robinson bound can be derived for arbitrary bounded observables. This is explained in Section 2. The novelty of this paper concerns harmonic and anharmonic lattice systems which have unbounded interactions of the form $(q_x - q_y)^2$. In Section 3 and Section 4 we prove Lieb-Robinson bounds for Weyl operators. The main advantage of working in the Weyl algebra is a consequence of the fact that the class of Weyl operators is invariant under the dynamics of the harmonic lattice model, a property that is also used in our treatment of anharmonic models. The Lieb-Robinson bounds that we obtain for the Weyl operators are sufficient to derive bounds for more general observables, such as q_x and p_x as well as compactly supported smooth bounded functions of q_x and p_x. This is discussed in Section 5.

Note that locality bounds for harmonic and anharmonic lattice systems have already been obtained in the classical setting; while harmonic systems are well-understood, anharmonic lattice systems are much more complicated, and a full understanding, even in the classical case, has not been reached, yet. In [17], Marchioro, Pellegrinotti, Pulvirenti, and Triolo considered anharmonic systems in thermal equilibrium and proved that, after time t, the influence of local perturbations becomes negligible at distances larger than $t^{4/3}$. These bounds were recently improved in [8] by Buttà, Caglioti, Di Ruzza, and Marchioro, who proved that after time t

local perturbations of thermal equilibrium are exponentially small in $\log^2 t$ at distances larger than $t \log^{\alpha} t$.

In the quantum mechanical setting, on the other hand, we are only aware of the recent work of Buerschaper, who derived, in [7], Lieb-Robinson type bounds for harmonic lattice systems.

2 Lieb-Robinson Estimates for Hamiltonians with Bounded Non-Local Terms

In this section, we will state and prove our first example of Lieb-Robinson estimates for systems with unbounded Hamiltonians. We consider here the dynamics generated by unbounded Hamiltonians, assuming, however, the unbounded interactions to be completely local. It turns out that, for such systems, locality bounds can be proven in the same generality as for quantum spin systems (see Theorem 2.1 below). Moreover, the proof of this result only requires minor modifications with respect to the arguments presented in [19].

We first introduce the underlying structure on which our models will be defined. Let Γ be an arbitrary set of sites equipped with a metric d. For Γ with infinite cardinality, we will need to assume that there exists a non-increasing function $F : [0, \infty) \to (0, \infty)$ for which:

i) F is uniformly integrable over Γ, i.e.,

$$\| F \| \, := \, \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) \, < \, \infty, \tag{2.1}$$

and

ii) F satisfies

$$C \, := \, \sup_{x,y \in \Gamma} \sum_{z \in \Gamma} \frac{F\left(d(x,z)\right) \, F\left(d(z,y)\right)}{F\left(d(x,y)\right)} \, < \, \infty. \tag{2.2}$$

Given such a set Γ and a function F, it is easy to see that for any $a \geq 0$ the function

$$F_a(x) = e^{-ax} \, F(x),$$

also satisfies i) and ii) above with $\|F_a\| \leq \|F\|$ and $C_a \leq C$.

In typical examples, one has that $\Gamma \subset \mathbb{Z}^{\nu}$ for some integer $\nu \geq 1$, and the metric is just given by $d(x,y) = |x-y| = \sum_{j=1}^{\nu} |x_j - y_j|$. In this case, the function F can be chosen as $F(|x|) = (1+|x|)^{-\nu-\epsilon}$ for any $\epsilon > 0$.

To each $x \in \Gamma$, we will associate a Hilbert space $\mathcal{H}_x$. Unlike in the setting of quantum spin systems, we will not assume that these Hilbert spaces are finite dimensional. For example,

in many relevant systems, one considers $\mathcal{H}_x = L^2(\mathbb{R}, \mathrm{d}q_x)$. With any finite subset $\Lambda \subset \Gamma$, the Hilbert space of states over Λ is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x,$$

and the local algebra of observables over Λ is then defined to be

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x),$$

where $\mathcal{B}(\mathcal{H}_x)$ denotes the algebra of bounded linear operators on $\mathcal{H}_x$.

If $\Lambda_1 \subset \Lambda_2$, then there is a natural way of identifying $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$, and (also in the case of infinite Γ) we may therefore define the algebra of local observables by the inductive limit

$$\mathcal{A}_\Gamma = \bigcup_{\Lambda\subset\Gamma} \mathcal{A}_\Lambda,$$

where the union is over all finite subsets $\Lambda \subset \Gamma$; see [4, 5] for a general discussion of these topics.

For the locality results we wish to describe, the notion of support of an observable will be important. The support of an observable $A \in \mathcal{A}_\Lambda$ is the minimal set $X \subset \Lambda$ for which $A = A' \otimes \mathbb{1}$ for some $A' \in \mathcal{A}_X = \bigotimes_{x\in X} \mathcal{B}(\mathcal{H}_x)$.

The result discussed in this section corresponds to bounded perturbations of local self-adjoint Hamiltonians. We fix a collection of local operators $H^{\mathrm{loc}} = \{H_x\}_{x\in\Gamma}$ where each H_x is a self-adjoint operator over $\mathcal{H}_x$. Again, we stress that these operators H_x need *not* be bounded.

In addition, we will consider a general class of bounded perturbations. These are defined in terms of an interaction Φ, which is a map from the set of subsets of Γ to $\mathcal{A}_\Gamma$ with the property that for each finite set $X \subset \Gamma$, $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X)^* = \Phi(X)$. To obtain our bound, we need to impose a growth restriction on the set of interactions Φ we consider. For any $a \geq 0$, denote by $\mathcal{B}_a(\Gamma)$ the set of interactions for which

$$\|\Phi\|_a := \sup_{x,y\in\Gamma} \frac{1}{F_a(d(x,y))} \sum_{X\ni x,y} \|\Phi(X)\| < \infty. \tag{2.3}$$

Now, for a fixed sequence of local Hamiltonians $H^{\mathrm{loc}} = \{H_x\}$, as described above, an interaction $\Phi \in \mathcal{B}_a(\Gamma)$, and a finite subset $\Lambda \subset \Gamma$, we will consider self-adjoint Hamiltonians of the form

$$H_\Lambda = H_\Lambda^{\mathrm{loc}} + H_\Lambda^{\Phi} = \sum_{x\in\Lambda} H_x + \sum_{X\subset\Lambda} \Phi(X), \tag{2.4}$$

acting on $\mathcal{H}_\Lambda$ (with domain given by $\bigotimes_{x\in\Lambda} D(H_x)$ where $D(H_x) \subset \mathcal{H}_x$ denotes the domain of H_x). As these operators are self-adjoint, they generate a dynamics, or time evolution, $\{\tau_t^\Lambda\}$, which is the one parameter group of automorphisms defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A\, e^{-itH_\Lambda} \quad \text{for any} \quad A \in \mathcal{A}_\Lambda.$$

For Hamiltonians of the form (2.4), we have a bound analogous to (1.1), see Theorem 2.1 below.

Before we present this result, we make an observation. It seems intuitively clear that the spread of interactions through a system should depend on the surface area of the support of the local observables being evolved; not their volume. One can make this explicit by introducing the following notation. Denote the surface of a set X, regarded as a subset of $\Lambda \subset \Gamma$, by

$$S_\Lambda(X) \,=\, \{ Z \subset \Lambda \,:\, Z \cap X \neq \emptyset \text{ and } Z \cap X^c \neq \emptyset \} \,. \tag{2.5}$$

Here we will use the notation $S(X) = S_\Gamma(X)$, and define the Φ-boundary of a set X, written $\partial_\Phi X$, by

$$\partial_\Phi X \,=\, \{ x \in X \,:\, \exists Z \in S(X) \text{ with } x \in Z \text{ and } \Phi(Z) \neq 0 \,\} \,.$$

We have the following result.

Theorem 2.1. *Fix a local Hamiltonian H^{loc} and an interaction $\Phi \in \mathcal{B}_a(\Gamma)$ for some $a \geq 0$. Let X and Y be subsets of Γ. Then, for any $\Lambda \supset X \cup Y$ and any pair of local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, one has that*

$$\left\| [\tau_t^\Lambda(A), B] \right\| \,\leq\, \frac{2\,\|A\|\,\|B\|}{C_a}\, g_a(t)\, D_a(X,Y), \tag{2.6}$$

where

$$g_a(t) = \begin{cases} e^{2\|\Phi\|_a C_a |t|} - 1 & \text{if } d(X,Y) > 0, \\ e^{2\|\Phi\|_a C_a |t|} & \text{otherwise,} \end{cases} \tag{2.7}$$

and $D_a(X,Y)$ is given by

$$D_a(X,Y) = \min\left[\sum_{x \in \partial_\Phi X} \sum_{y \in Y} F_a\left(d(x,y)\right), \sum_{x \in X} \sum_{y \in \partial_\Phi Y} F_a\left(d(x,y)\right) \right]. \tag{2.8}$$

The following corollary provides a bound in terms of $d(X,Y) = \min_{x \in X, y \in Y} d(x,y)$, the distance between the supports X, Y.

Corollary 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left\| [\tau_t^\Lambda(A), B] \right\| \,\leq\, \frac{2\,\|A\|\,\|B\|\,\|F\|}{C_a} \min\left[|\partial_\Phi X| \,, |\partial_\Phi Y| \right] \, e^{-a\left[d(X,Y) - \frac{2\|\Phi\|_a C_a}{a} |t| \right]}, \tag{2.9}$$

Proof of Theorem 2.1. For any finite $Z \subset \Gamma$, we introduce the quantity

$$C_B(Z;t) \,:=\, \sup_{A \in \mathcal{A}_Z} \frac{\| [\tau_t^\Lambda(A), B] \|}{\|A\|}, \tag{2.10}$$

and note that $C_B(Z;0) \leq 2\|B\| \delta_Y(Z)$, where we defined $\delta_Y(Z) = 1$ if $Y \cap Z \neq \emptyset$ and $\delta_Y(Z) = 0$ if $Y \cap Z = \emptyset$. A key observation in our proof will be the fact that the dynamics generated by

$$H_\Lambda^{\mathrm{loc}} \,+\, H_X^\Phi \,=\, \sum_{x \in \Lambda} H_x \,+\, \sum_{Z \subset X} \Phi(Z)$$

remains local. More precisely, if we define

$$\tau_t^{\mathrm{loc}}(A) \;=\; e^{it\left(H_\Lambda^{\mathrm{loc}}+H_X^\Phi\right)}\, A\, e^{-it\left(H_\Lambda^{\mathrm{loc}}+H_X^\Phi\right)} \quad \text{for all} \quad A \in \mathcal{A}_\Lambda, \tag{2.11}$$

we have that for every $A \in \mathcal{A}_X$, $\tau_t^{\mathrm{loc}}(A) \in \mathcal{A}_X$ for every $t \in \mathbb{R}$. This implies, recalling the definition (2.10), that

$$C_B(X;t) \;=\; \sup_{A\in\mathcal{A}_X} \frac{\|[\tau_t^\Lambda(\tau_{-t}^{\mathrm{loc}}(A)), B]\|}{\|A\|}\,. \tag{2.12}$$

Consider the function (setting $\tau_t(\cdot) = \tau_t^\Lambda(\cdot)$)

$$f(t) \;:=\; \left[\tau_t\left(\tau_{-t}^{\mathrm{loc}}(A)\right), B\right],$$

for $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$. It is straightforward to verify that

$$f'(t) = i \sum_{Z\in S_\Lambda(X)} \left[\tau_t\left(\Phi(Z)\right), f(t)\right] - i \sum_{Z\in\, S_\Lambda(X)} \left[\tau_t(\tau_{-t}^{\mathrm{loc}}(A)), \left[\tau_t\left(\Phi(Z)\right), B\right]\right]. \tag{2.13}$$

As is discussed in [19, Appendix A], the first term in the above differential equation is norm preserving, and therefore we have the bound

$$\|f(t)\| \;\leq\; \|f(0)\| \;+\; 2\|A\| \sum_{Z\in S(X)} \int_0^{|t|} \|[\tau_s(\Phi(Z)), B]\| ds. \tag{2.14}$$

Recalling definition (2.10), the above inequality readily implies that

$$C_B(X,t) \leq C_B(X,0) + 2 \sum_{Z\in\, S(X)} \|\Phi(Z)\| \int_0^{|t|} C_B(Z,s) ds, \tag{2.15}$$

where we have used (2.12). Iterating this inequality, exactly as is done in [19], see also [21], yields (2.6) with (2.7) and (2.8). The inequality (2.9), stated in the corollary, readily follows. □

In many situations, $\Lambda \subset \mathbb{Z}^\nu$ and the bound (2.9) can be made slightly more explicit (but less optimal) by choosing

$$F(x) = (1+|x|)^{-\nu-1}, \quad \text{and } C = 2^{\nu+1} \sum_{x\in\mathbb{Z}^\nu} \frac{1}{(1+|x|)^{\nu+1}}\,.$$

In this case we have

$$\|[\tau_t^\Lambda(A), B]\| \;\leq\; 2^{-(\nu+1)}\, \|A\|\|B\|\, \min[|\partial_\Phi X|, |\partial_\Phi Y|]\, e^{-(ad(X,Y)-2\|\Phi\|_a C|t|)}. \tag{2.16}$$

for all $a > 0$, with

$$\|\Phi\|_a = \sup_{x,y\in\Lambda} e^{a|x-y|}(1+|x-y|)^{\nu+1} \sum_{X\ni x,y} \|\Phi(X)\| < \infty\,.$$

Eq. (2.16) gives the upper bound $2\|\Phi\|_a C/a$ for the speed of propagation in these systems.

One application of the general framework used in Theorem 2.1 concerns systems comprised of finite clusters with possibly unbounded interactions within each cluster but only bounded interactions between clusters. For such systems, by adjusting Γ and $d(x,y)$, Theorem 2.1 still applies.

3 Harmonic Lattice Systems

In this section, we present our second example of Lieb-Robinson bounds for systems with unbounded Hamiltonians. Let L and ν be positive integers. We will consider harmonic Hamiltonians defined on cubic subsets $\Lambda_L = (-L, L]^\nu \cap \mathbb{Z}^\nu$. Specifically, for $j = 1, \ldots, \nu$ and real parameters $\lambda_j \geq 0$ and $\omega > 0$, we will analyze the Hamiltonian

$$H_L^h \;=\; H_L^h(\{\lambda_j\}, \omega) \;=\; \sum_{x \in \Lambda_L} p_x^2 \,+\, \omega^2\, q_x^2 \,+\, \sum_{j=1}^{\nu} \lambda_j\, (q_x - q_{x+e_j})^2, \tag{3.1}$$

with periodic boundary conditions (in the sense that $q_{x+e_j} := q_{x-(2L-1)e_j}$ if $x \in \Lambda_L$ but $x + e_j \notin \Lambda_L$), acting in the Hilbert space

$$\mathcal{H}_{\Lambda_L} = \bigotimes_{x \in \Lambda_L} L^2(\mathbb{R}, dq_x). \tag{3.2}$$

Here $\{e_j\}_{j=1}^{\nu}$ are the canonical basis vectors in $\mathbb{Z}^\nu$, and since, in most calculations, the values of λ_j and ω will be fixed, we will simply write H_L^h for notational convenience. The quantities p_x and q_x, which appear in (3.1) above, are the single site momentum and position operators regarded as operators on the full Hilbert space $\mathcal{H}_{\Lambda_L}$ by setting (we use here units with $\hbar = 1$)

$$p_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes -i\frac{d}{dq} \otimes \mathbb{1} \cdots \otimes \mathbb{1} \quad \text{and} \quad q_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes q \otimes \mathbb{1} \cdots \otimes \mathbb{1}, \tag{3.3}$$

i.e., these operators act non-trivially only in the x-th factor of $\mathcal{H}_{\Lambda_L}$. These operators satisfy the canonical commutation relations

$$[p_x, p_y] \;=\; [q_x, q_y] \;=\; 0 \quad \text{and} \quad [q_x, p_y] \;=\; i\delta_{x,y}, \tag{3.4}$$

valid for all $x, y \in \Lambda_L$. The Hamiltonian H_L^h describes a system of coupled harmonic oscillators (with mass $m = 1/2$) sitting at all $x \in \Lambda_L$.

Let $\mathcal{A}_{\Lambda_L}$ be the algebra of all bounded observables on $\mathcal{H}_{\Lambda_L}$. The time-evolution generated by the Hamiltonian (3.1) is the one-parameter group of automorphisms $\{\tau_t^{h;\Lambda_L}\}_{t \in \mathbb{R}}$ of $\mathcal{A}_{\Lambda_L}$, defined by

$$\tau_t^{h;\Lambda_L}(A) = e^{itH_L^h} A e^{-itH_L^h}. \tag{3.5}$$

As we will regard the length scale L to be fixed, we will suppress the dependence of the dynamics on Λ_L in our notation, by setting $\tau_t^h(.) = \tau_t^{h;\Lambda_L}$.

An important class of observables in $\mathcal{A}_{\Lambda_L}$ are the Weyl operators. For a bounded, complex-valued function $f : \Lambda \to \mathbb{C}$, we define the Weyl operator $W(f)$ by

$$W(f) = e^{i \sum_{x \in \Lambda} (q_x \text{Re } f_x + p_x \text{Im } f_x)} \tag{3.6}$$

Clearly, $W(f)$ is a unitary operator in $\mathcal{A}_{\Lambda_L}$ such that

$$W^{-1}(f) = W^*(f) = W(-f)\,.$$

Moreover, using the well-known Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{if} \quad [A,[A,B]] = [B,[A,B]] = 0, \tag{3.7}$$

and the commutation relations (3.4), it follows that Weyl operators satisfy the Weyl relations

$$W(f)\,W(g) \;=\; W(g)\,W(f)\,e^{-i\mathrm{Im}[\langle f,g\rangle]} \;=\; W(f+g)\,e^{-\frac{i}{2}\mathrm{Im}[\langle f,g\rangle]} \tag{3.8}$$

for any bounded $f, g : \Lambda \to \mathbb{C}$, and that they generate shifts of the position and the momentum operator, in the sense that

$$W^*(f)\,q_x\,W(f) = q_x - \mathrm{Im}\ f_x \qquad \text{and}\ W^*(f)\,p_x\,W(f) = p_x + \mathrm{Re}\ f_x\,. \tag{3.9}$$

The main result of this section is a Lieb-Robinson bound for the harmonic time-evolution of Weyl operators.

Theorem 3.1. *For any finite $X, Y \subset \mathbb{Z}^\nu$, for all $L > 0$ such that $X, Y \subset \Lambda_L$, and for any functions f and g with $\mathrm{supp}(f) \subset X$ and $\mathrm{supp}(g) \subset Y$, the estimate*

$$\left\| \left[\tau_t^h\left(W(f)\right), W(g) \right] \right\| \;\leq\; C\,\|f\|_\infty \|g\|_\infty \sum_{x\in X, y\in Y} e^{-\mu\left(d(x,y) - c_{\omega,\lambda}\max\left(\frac{2}{\mu},\, e^{(\mu/2)+1}\right)|t|\right)} \tag{3.10}$$

holds for all $\mu > 0$. Here

$$d(x,y) = \sum_{j=1}^{\nu} \min_{\eta_j \in \mathbb{Z}} |x_j - y_j + 2L\eta_j|\,. \tag{3.11}$$

is the distance on the torus. Moreover

$$C = \left(2 + c_{\omega,\lambda} e^{\mu/2} + c_{\omega,\lambda}^{-1}\right) \tag{3.12}$$

with $c_{\omega,\lambda} = (\omega^2 + 4\sum_{j=1}^{\nu}\lambda_j)^{1/2}$.

Corollary 3.2. *Under the same conditions as in Theorem 3.1, for any $0 < a < 1$, one has*

$$\left\| \left[\tau_t^h\left(W(f)\right), W(g) \right] \right\| \;\leq\; \tilde{C}\,\|f\|_\infty \|g\|_\infty\ \min(|X|, |Y|)\, e^{-\mu\left(a d(X,Y) - c_{\omega,\lambda}\max\left(\frac{2}{\mu},\, e^{(\mu/2)+1}\right)|t|\right)} \tag{3.13}$$

where

$$d(X,Y) = \min_{x\in X, y\in Y} d(x,y)$$

and

$$\tilde{C} = C \sum_{z\in\mathbb{Z}^\nu} e^{-\mu(1-a)|z|}\,.$$

Remark 3.3. *i) As we will discuss in Remark 3.6 (see also Lemma 3.7), both Theorem 3.1 and Corollary 3.2 remain valid in the case* $\omega = 0$.

ii) If we make the further assumption that the sets X *and* Y *have a minimal separation distance, then a stronger, "small time" version of (3.10) holds. Specifically, let* $\mu > 0$ *be given, and assume that* X *and* Y *have been chosen with* $d(X,Y) > 1 + c_{\omega,\lambda} e^{(\mu/2)+1}$. *Then for any functions* f *and* g *with support in* X *and* Y, *respectively, one has that*

$$\left\| \left[\tau_t^h \left(W(f) \right), W(g) \right] \right\| \leq t^{2d(X,Y)} \, C \, \|f\|_\infty \|g\|_\infty \sum_{x \in X, y \in Y} e^{-\mu \left(d(x,y) - c_{\omega,\lambda} \max\left(\frac{2}{\mu}, e^{(\mu/2)+1} \right) |t| \right)}. \tag{3.14}$$

This bound follows from factoring the $t^{2|x|}$ *out of (3.43), and then completing the argument as before.*

iii) In most applications of the Lieb-Robinson bound it is important to obtain an estimate on the group velocity, referred to as the Lieb-Robinson velocity [18, 14, 19, 6, 10, 13, 21]. Note that we can obtain arbitrarily fast exponential decay in space at the cost of a worse estimate for the Lieb-Robinson velocity:

$$v_h(\mu) = c_{\omega,\lambda} \max\left(\frac{2}{\mu}, e^{(\mu/2)+1}\right). \tag{3.15}$$

The optimal Lieb-Robinson velocity in the above estimates is obtained by choosing $\mu = \mu_0$, *the solution of*

$$\frac{2}{\mu} = e^{(\mu/2)+1}.$$

Clearly, $1/2 < \mu_0 < 1$. *This gives the following bound for the Lieb-Robinson velocity in the harmonic lattice:* $v_h(\mu_0) = 2c_{\omega,\lambda}/\mu_0 \leq 4c_{\omega,\lambda}$.

Theorem 3.1 follows from Lemma 3.4 and Lemma 3.5, both proven below. In Lemma 3.4, we derive an explicit formula for the time evolution of a Weyl operator. This allows us to bound the norm on the l.h.s. of (3.10) by certain Fourier sums which we then estimate in Lemma 3.5.

For bounded functions $f, g : \Lambda_L \to \mathbb{C}$, we define the convolution $(f * g) : \Lambda_L \to \mathbb{C}$ by

$$(f * g)_x = \sum_{y \in \Lambda_L} f_y \, g_{x-y}, \tag{3.16}$$

for any $x \in \Lambda_L$ (if $(x-y) \notin \Lambda_L$, then we define g_{x-y} through the periodic boundary conditions).

Lemma 3.4. *Let* L *be a positive integer and consider a bounded function* $f : \Lambda_L \to \mathbb{C}$. *Then the harmonic evolution of the Weyl operator* $W(f)$ *is the Weyl operator given by*

$$\tau_t^h \left(W(f) \right) = W\left(f_t\right), \quad f_t = f * \overline{h_{1,t}^{(L)}} + \overline{f} * h_{2,t}^{(L)}. \tag{3.17}$$

Here the even functions $h_{1,t}$ and $h_{2,t}$ are given by

$$h_{1,t}^{(L)}(x) \;=\; \frac{i}{2}\mathrm{Im}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*}\left(\gamma(k) \,+\, \frac{1}{\gamma(k)}\right)\, e^{ik\cdot x-2i\gamma(k)t}\right] \,+\, \mathrm{Re}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} e^{ik\cdot x-2i\gamma(k)t}\right], \tag{3.18}$$

and

$$h_{2,t}^{(L)}(x) \;=\; \frac{i}{2}\mathrm{Im}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*}\left(\gamma(k) \,-\, \frac{1}{\gamma(k)}\right)\, e^{ik\cdot x-2i\gamma(k)t}\right], \tag{3.19}$$

where

$$\Lambda_L^* = \left\{\frac{x\pi}{L} \;:\; x\in\Lambda_L\right\}$$

and

$$\gamma(k;\omega,\{\lambda_j\}) \;=\; \gamma(k) \;=\; \sqrt{\omega^2 \,+\, 4\sum_{j=1}^{\nu}\lambda_j\,\sin^2(k_j/2)}. \tag{3.20}$$

The proof of Lemma 3.4 is given in Section 3.1.

Lemma 3.5. *Suppose that the functions $h_{1,t}^{(L)}, h_{2,t}^{(L)} : \Lambda_L \to \mathbb{C}$ are defined as in (3.18), (3.19). Then*

$$|h_{m,t}^{(L)}(x)| \le \left(1+\frac{1}{2}c_{\omega,\lambda}e^{\mu/2}+\frac{1}{2}c_{\omega,\lambda}^{-1}\right)\, e^{-\mu\left(|x|-c_{\omega,\lambda}\max\left(\frac{2}{\mu},\,e^{(\mu/2)+1}\right)|t|\right)}$$

for $m=1,2$, all $\mu>0$, $t\in\mathbb{R}$, and $x\in\Lambda_L$. Here we defined $c_{\omega,\lambda}=(\omega^2+4\sum_{j=1}^{\nu}\lambda_j)^{1/2}$ and $|x|=\sum_{j=1}^{\nu}|x_i|$. Note that the bounds are uniform in L.

The proof of Lemma 3.5 can be found in Section 3.2. Using these two lemmas, we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let f and g be functions supported in disjoint sets X and Y, respectively, with separation distance $d(X,Y)>0$. Let $L>0$ be large enough so that $X\cup Y\subset\Lambda_L$. With Lemma 3.4 and the Weyl relations (3.8), it is clear that

$$\left[\tau_t^h\left(W(f)\right),\, W(g)\right] \;=\; W\left(f_t\right)\,W(g)\,\left(1\,-\,e^{-i\mathrm{Im}[\langle g,f_t\rangle]}\right).$$

Using the above formula, it follows that

$$\left\|\left[\tau_t^h\left(W(f)\right),\, W(g)\right]\right\| \;\le\; |\,\mathrm{Im}\left[\langle g,f_t\rangle\right]| \;\le\; \left|\langle g, f*\overline{h_{1,t}^{(L)}} \,+\, \overline{f}*h_{2,t}^{(L)}\rangle\right|. \tag{3.21}$$

Expanding the first term, we find that

$$\langle g, f*\overline{h_{1,t}^{(L)}}\rangle \;=\; \sum_{y\in\Lambda_L}\overline{g_y}\,\left(f*h_{1,t}^{(L)}\right)_y \;=\; \sum_{y\in Y}\sum_{x\in X}\overline{g_y}\,f_x\,\overline{h_{1,t}^{(L)}(y-x)}, \tag{3.22}$$

and therefore the bound

$$
\begin{aligned}
& \big|\,\langle\, g,\, f * \overline{h_{1,t}^{(L)}}\,\rangle\,\big| \\
& \qquad \leq \|f\|_\infty\,\|g\|_\infty \sum_{x\in X, y\in Y} \left| h_{1,t}^{(L)}(x-y) \right| \\
& \qquad \leq \left(1+\frac{1}{2}c_{\omega,\lambda}e^{\mu/2}+\frac{1}{2}c_{\omega,\lambda}^{-1}\right)\,\|f\|_\infty\,\|g\|_\infty \sum_{x\in X, y\in Y} e^{-\mu\left(d(x,y)-c_{\omega,\lambda}\max\left(\frac{2}{\mu},\, e^{(\mu/2)+1}\right)|t|\right)}
\end{aligned}
\tag{3.23}
$$

follows from Lemma 3.5. A similar analysis applies to the second term on the r.h.s. of (3.21), yielding (3.10). □

3.1 Harmonic Evolution of Weyl Operators

The goal of this section is to prove Lemma 3.4. To this end, we diagonalize the harmonic Hamiltonian H_L^h by introducing Fourier space operators. Consider the set (recall that $\Lambda_L = (-L, L]^\nu \cap \mathbb{Z}^\nu$)

$$
\Lambda_L^* \,=\, \left\{ \frac{x\pi}{L} \,:\, x \in \Lambda_L \right\}.
$$

Then it is clear that $\Lambda_L^* \subset (-\pi, \pi]^\nu$ and $|\Lambda_L^*| = (2L)^\nu = |\Lambda_L|$. For each $k \in \Lambda_L^*$, we introduce the operators,

$$
Q_k \,=\, \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x\in\Lambda_L} e^{-ik\cdot x} q_x \quad \text{and} \quad P_k \,=\, \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x\in\Lambda_L} e^{-ik\cdot x} p_x \,. \tag{3.24}
$$

One may easily calculate that $Q_k^* = Q_{-k}$ (similarly, $P_k^* = P_{-k}$) for all $k \in \Lambda_L^*$. Here we have adopted the convention that for $k = (k_1, \ldots, k_\nu) \in \Lambda_L^*$, $-k$ is defined to be the element of Λ_L^* whose components are given by

$$
(-k)_j \,=\, \begin{cases} -k_j, & \text{if } |k_j| < \pi, \\ \pi, & \text{otherwise.} \end{cases}
$$

This is reasonable as $e^{i\pi x} = e^{-i\pi x}$ for all integers x. These operators satisfy the following commutation relations

$$
[Q_k, Q_{k'}] \,=\, [P_k, P_{k'}] \,=\, 0 \quad \text{and} \quad [Q_k, P_{k'}] \,=\, i\,\delta_{k,-k'}, \tag{3.25}
$$

for any $k, k' \in \Lambda_L^*$. Furthermore, for any $x \in \Lambda_L$,

$$
q_x \,=\, \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} e^{ik\cdot x} Q_k \quad \text{and} \quad p_x \,=\, \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} e^{ik\cdot x} P_k \,. \tag{3.26}
$$

With the above relations, it is easy to check that the harmonic Hamiltonian (3.1) can be rewritten as

$$
H_L^h \,=\, \sum_{k\in\Lambda_L^*} P_k P_{-k} \,+\, \gamma^2(k) Q_k Q_{-k} \,. \tag{3.27}
$$

where we introduced the notation

$$\gamma(k) \,=\, \gamma(k;\{\lambda_j\},\omega) \,=\, \sqrt{\omega^2 \,+\, 4\sum_{j=1}^{\nu} \lambda_j \, \sin^2(k_j/2)}\,. \tag{3.28}$$

Observe that $\gamma(k)$ is independent of sign changes in any component of k.

Since we have assumed that $\omega > 0$, we have that $\gamma(k) \geq \omega > 0$, and therefore, we may diagonalize the Hamiltonian by setting

$$b_k \,=\, \frac{1}{\sqrt{2\gamma(k)}}\, P_k - i\sqrt{\frac{\gamma(k)}{2}}\, Q_k \quad \text{and} \quad b_k^* \,=\, \frac{1}{\sqrt{2\gamma(k)}}\, P_{-k} + i\sqrt{\frac{\gamma(k)}{2}}\, Q_{-k}\,. \tag{3.29}$$

In fact, as a result of this definition, we find that for $k, k' \in \Lambda_L^*$

$$[b_k, b_{k'}] \,=\, [b_k^*, b_{k'}^*] \,=\, 0 \quad \text{and} \quad [b_k, b_{k'}^*] \,=\, \delta_{k,k'}, \tag{3.30}$$

and moreover, for each $k \in \Lambda_L^*$,

$$Q_k \,=\, \frac{i}{\sqrt{2\gamma(k)}}\left(b_k \,-\, b_{-k}^*\right) \quad \text{and} \quad P_k \,=\, \sqrt{\frac{\gamma(k)}{2}}\left(b_k \,+\, b_{-k}^*\right). \tag{3.31}$$

Inserting the above into (3.27), we have that

$$H_L^h \,=\, \sum_{k\in\Lambda_L^*} \gamma(k)\; (\, 2\, b_k^*\, b_k \,+\, 1\,)\,. \tag{3.32}$$

From this representation of the Hamiltonian H_L^h, we obtain immediately the Heisenberg evolution of the operators b_k and b_k^*. In fact, from the commutation relations (3.30), it follows that

$$\tau_t^h(b_k) \,=\, e^{-2i\gamma(k)t}\, b_k \quad \text{and} \quad \tau_t^h(b_k^*) \,=\, e^{2i\gamma(k)t}\, b_k^* \tag{3.33}$$

for all $t \in \mathbb{R}$.

To compute the evolution of the operators p_x and q_x, for $x \in \Lambda_L$, we express them in terms of b_k and b_k^*. We find

$$\begin{aligned} q_x &= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} e^{ik\cdot x} Q_k = \frac{i}{\sqrt{2|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \frac{e^{ik\cdot x}}{\sqrt{\gamma(k)}}\left(b_k - b_{-k}^*\right) \\ p_x &= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} e^{ik\cdot x} P_k = \frac{1}{\sqrt{2|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \sqrt{\gamma(k)}\; e^{ik\cdot x}\; \left(b_k + b_{-k}^*\right)\,. \end{aligned} \tag{3.34}$$

Therefore

$$\begin{aligned} \tau_t^h(q_x) &= \frac{i}{\sqrt{2|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \frac{e^{ik\cdot x}}{\sqrt{\gamma(k)}}\left(e^{-2i\gamma(k)t}\, b_k - e^{2i\gamma(k)t} b_{-k}^*\right) \\ &= \frac{i}{\sqrt{2|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \frac{1}{\sqrt{\gamma(k)}}\left(e^{ik\cdot x-2i\gamma(k)t}\, b_k - e^{-ik\cdot x+2i\gamma(k)t}\, b_k^*\right) \end{aligned}$$

and

$$\tau_t^h(p_x) = \frac{1}{\sqrt{2|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \sqrt{\gamma(k)} \, \left(e^{ik\cdot x-2i\gamma(k)t}\, b_k + e^{-ik\cdot x+2i\gamma(k)t}\, b_k^*\right) .$$

From (3.29) and (3.26), it follows that

$$\begin{aligned}\tau_t^h(q_x) \, &= \frac{i}{2|\Lambda_L|} \sum_{k\in\Lambda_L^*} \frac{e^{ik\cdot x-2i\gamma(k)t}}{\sqrt{\gamma(k)}} \left(\frac{1}{\sqrt{\gamma(k)}} \sum_{y\in\Lambda_L} e^{-ik\cdot y} p_y - i\sqrt{\gamma(k)} \sum_{y\in\Lambda_L} e^{-ik\cdot y} q_y \right) \\ &\quad - \frac{i}{2|\Lambda_L|} \sum_{k\in\Lambda_L^*} \frac{e^{-ik\cdot x+2i\gamma(k)t}}{\sqrt{\gamma(k)}} \left(\frac{1}{\sqrt{\gamma(k)}} \sum_{y\in\Lambda_L} e^{ik\cdot y} p_y + i\sqrt{\gamma(k)} \sum_{y\in\Lambda_L} e^{ik\cdot y} q_y \right)\end{aligned}$$

which implies

$$\tau_t^h(q_x) \, = \sum_{y\in\Lambda_L} q_y \, \mathrm{Re} \, \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L^*} e^{ik\cdot(x-y)-2i\gamma(k)t} - \sum_{y\in\Lambda_L} p_y \, \mathrm{Im} \, \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L^*} \frac{1}{\gamma(k)} e^{ik\cdot(x-y)-2i\gamma(k)t} .$$

Analogously, we find

$$\tau_t^h(p_x) \, = \sum_{y\in\Lambda_L} p_y \, \mathrm{Re} \, \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L^*} e^{ik\cdot(x-y)-2i\gamma(k)t} + \sum_{y\in\Lambda_L} q_y \, \mathrm{Im} \, \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L^*} \gamma(k) e^{ik\cdot(x-y)-2i\gamma(k)t} .$$

It is then easy to check that

$$\tau_t^h \left(\sum_{x\in\Lambda_L} q_x \, \mathrm{Re} \, f_x + p_x \, \mathrm{Im} \, f_x \right) = \sum_{x\in\Lambda_L} q_x \mathrm{Re} \, (f_t)_x + p_x \mathrm{Im} \, (f_t)_x$$

with

$$f_t = f * \overline{h_{1,t}^{(L)}} + \overline{f} * h_{2,t}^{(L)}$$

and where $h_{1,t}^{(L)}$ and $h_{2,t}^{(L)}$ are defined as in (3.18), (3.19). This proves (3.17).

Remark 3.6. *If we consider the Hamiltonian (3.1) with $\omega = 0$, then we can easily obtain analogous formulas for the time evolution of Weyl operators. In fact, if $\omega = 0$, we can still define operators P_k, Q_k as in (3.24) and, for every $k \in \Lambda_L^* \backslash \{0\}$, operators b_k and b_k^* exactly as in (3.30). In terms of these operators, the Hamiltonian (3.1) can be expressed, in the case $\omega = 0$, as*

$$H_L^h (\omega = 0) = \, P_0^2 \, + \sum_{k\in\Lambda_L^*\backslash\{0\}} \gamma(k) \, \left(\, 2\, b_k^* \, b_k \, + \, 1 \, \right) .$$

Since P_0 commutes with b_k, b_k^*, for all $k \neq 0$, we obtain (using the commutation relation (3.30) and (3.25)) that

$$\begin{aligned} \tau_t^h(b_k) \, &= \, e^{-2i\gamma(k)t} \, b_k, \qquad & \tau_t^h(b_k^*) \, &= \, e^{2i\gamma(k)t} \, b_k^*, \\ \tau_t^h(P_0) \, &= \, P_0, \qquad \textit{and} & \tau_t^h(Q_0) &= Q_0 + 2tP_0 \, . \end{aligned}$$

From these formulae, we find that, in the case $\omega = 0$,

$$\tau_t^h \left(W(f) \right) \, = \, W \left(\, f * \overline{h_{0,1,t}^{(L)}} \, + \, \overline{f} * h_{0,2,t}^{(L)} \right),$$

with

$$h^{(L)}_{0,1,t}(x) \;=\; \frac{(1-it)}{|\Lambda_L|} \;+\; \tilde{h}^{(L)}_{1,t}(x),$$
$$h^{(L)}_{0,2,t}(x) \;=\; \frac{it}{|\Lambda_L|} \;+\; \tilde{h}^{(L)}_{2,t}(x).$$

and where

$$\begin{aligned}\tilde{h}^{(L)}_{1,t}(x) \;=&\frac{i}{2}\mathrm{Im}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L\setminus\{k_0\}}\left(\gamma(k)\;+\;\frac{1}{\gamma(k)}\right)\,e^{ik\cdot x-2i\gamma(k)t}\right]\\ &+\;\mathrm{Re}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L\setminus\{k_0\}}e^{ik\cdot x-2i\gamma(k)t}\right],\end{aligned} \tag{3.35}$$

and

$$\tilde{h}^{(L)}_{2,t}(x) \;=\; \frac{i}{2}\mathrm{Im}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L\setminus\{k_0\}}\left(\gamma(k)\;-\;\frac{1}{\gamma(k)}\right)\,e^{ik\cdot x-2i\gamma(k)t}\right]. \tag{3.36}$$

3.2 Estimates on Fourier Sums. Proof of Lemma 3.5

The goal of this section is to prove Lemma 3.5. For $x\in\Lambda_L$, let

$$\begin{aligned}H^{(0)}_L(t,x) &= \mathrm{Re}\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L}e^{i\,k\cdot x\,-\,2\,i\,\gamma(k)\,t}\\ H^{(1)}_L(t,x) &= \mathrm{Im}\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L}\gamma(k)\;e^{i\,k\cdot x\,-\,2\,i\,\gamma(k)\,t}\\ H^{(-1)}_L(t,x) &= \mathrm{Im}\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda^*_L}\frac{1}{\gamma(k)}\;e^{i\,k\cdot x\,-\,2\,i\,\gamma(k)\,t}\,.\end{aligned} \tag{3.37}$$

Since $h^{(L)}_{1,t}(x) = H^{(0)}_L(t,x)+(i/2)(H^{(1)}_L(t,x)+H^{(-1)}_L(t,x))$ and $h^{(L)}_{2,t}(x) = (i/2)(H^{(1)}_L(t,x) - H^{(-1)}_L(t,x))$, Lemma 3.5 follows from the following exponential estimates on $H^{(m)}_L(t,x)$.

Lemma 3.7. *Suppose that $H^{(m)}_L(t,x)$, for $m=-1,1,0$, is defined as in (3.37), with $\gamma(k) = (\omega^2+4\sum_{j=1}^{\nu}\lambda_j\sin^2(k_j/2))^{1/2}$, and $\omega\geq 0$. Then we have*

$$\begin{aligned}|H^{(0)}_L(t,x)| &\leq\; e^{-\mu\left(|x|-c_{\omega,\lambda}\max\left(\frac{2}{\mu}\,,\,e^{(\mu/2)+1}\right)|t|\right)}\\ |H^{(1)}_L(t,x)| &\leq\; c_{\omega,\lambda}e^{\frac{\mu}{2}}\;e^{-\mu\left(|x|-c_{\omega,\lambda}\max\left(\frac{2}{\mu}\,,\,e^{(\mu/2)+1}\right)|t|\right)}\\ |H^{(-1)}_L(t,x)| &\leq\; c^{-1}_{\omega,\lambda}\;e^{-\mu\left(|x|-c_{\omega,\lambda}\max\left(\frac{2}{\mu}\,,\,e^{(\mu/2)+1}\right)|t|\right)}\end{aligned} \tag{3.38}$$

for all $\mu>0$, $x\in\Lambda_L$, $t\in\mathbb{R}$, and $L>0$. Here $c_{\omega,\lambda} = (\omega^2+4\sum_{j=1}^{\nu}\lambda_j)^{1/2}$.

Proof of Lemma 3.7. We first prove (3.38) for $m = 0$. Since $m = 0$ throughout this proof, and also L is fixed, we will use here the shorthand notation $H(t,x)$ for $H_L^{(0)}(t,x)$. We start by expanding the exponent $e^{-2i\gamma(k)t}$;

$$\begin{aligned} H(t,x) &= \mathrm{Re}\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} e^{ik\cdot x}\sum_{n\geq 0}\frac{(-2it\gamma(k))^n}{n!} \\ &= \mathrm{Re}\sum_{n\geq 0}\frac{(-1)^n 4^n t^{2n}}{(2n)!}\;\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} e^{ik\cdot x}\gamma^{2n}(k) \\ &\quad + 2\,\mathrm{Im}\sum_{n\geq 0}\frac{(-1)^n 4^n t^{2n+1}}{(2n+1)!}\;\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} e^{ik\cdot x}\gamma^{2n+1}(k)\,. \end{aligned}$$

The second term vanishes because $\gamma(-k) = \gamma(k)$. As for the first term we expand the exponent $\gamma^{2n}(k)$. We find

$$\begin{aligned} H(t,x) = \sum_{n\geq 0}\frac{(-1)^n 4^n t^{2n}}{(2n)!}\sum_{\substack{m_0,m_1,\dots,m_\nu\geq 0\\ m_0+\dots+m_\nu=n}}\frac{n!}{m_0!m_1!\dots m_\nu!}\,\omega^{2m_0} \\ \times\prod_{j=1}^{\nu}(4\lambda_j)^{m_j}\frac{1}{2L}\sum_{\substack{k_j=\frac{\pi}{L}\ell:\\ \ell=-L+1,\dots L}} e^{ik_jx_j}\sin^{2m_j}(k_j/2)\,. \end{aligned} \tag{3.39}$$

Next we note that, for $-L < x_j \leq L$,

$$\frac{1}{2L}\sum_{\substack{k_j=\frac{\pi}{L}\ell:\\ \ell=-L+1,\dots L}} e^{ik_jx_j}\sin^{2m_j}(k_j/2) = 0 \tag{3.40}$$

if $|x_j| > m_j$. This follows from the orthogonality relation

$$\frac{1}{2L}\sum_{\substack{k=\frac{\pi}{L}\ell:\\ \ell=-L+1,\dots L}} e^{ikx} = \delta_{x,0}$$

if $x \in \Lambda_L$, and from the observation that

$$\begin{aligned} e^{ik_jx_j}\sin^{2m_j}(k_j/2) &= e^{ik_jx_j}\frac{(1-\cos k_j)^{m_j}}{2^{m_j}} \\ &= \frac{1}{2^{m_j}}\sum_{\ell=0}^{m_j}\binom{m_j}{\ell}\frac{(-1)^\ell}{2^\ell}\sum_{p=0}^{\ell}\binom{\ell}{p}e^{i(x_j+2p-\ell)k_j}\,. \end{aligned} \tag{3.41}$$

Since $-m_j \leq -\ell \leq 2p-\ell \leq \ell \leq m_j$, we obtain (3.40). Since moreover

$$\Big|\frac{1}{2L}\sum_{\substack{k_j=\frac{\pi}{L}\ell:\\ \ell=-L+1,\dots L}} e^{ik_jx_j}\sin^{2m_j}(k_j/2)\Big| \leq 1$$

for all x_j and m_j, we obtain, from (3.39),

$$\begin{aligned} |H(t,x)| &\leq \sum_{n\geq|x|} \frac{4^n t^{2n}}{(2n)!} \sum_{\substack{m_0,m_1,\dots,m_\nu\geq 0\\ m_0+\dots+m_\nu=n}} \frac{n!}{m_0!m_1!\dots m_\nu!}\, \omega^{2m_0} \prod_{j=1}^{\nu} (4\lambda_j)^{m_j} \\ &= \sum_{n\geq|x|} \frac{(2c_{\omega,\lambda} t)^{2n}}{(2n)!} \end{aligned} \tag{3.42}$$

where we put $c_{\omega,\lambda} = (\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j)^{1/2}$. The previous inequality implies that

$$|H(t,x)| \leq \sum_{n\geq|x|} \frac{(2c_{\omega,\lambda}|t|)^{2n}}{(2n)!} \leq \frac{(2c_{\omega,\lambda}|t|)^{2|x|}}{(2|x|)!}\, e^{2c_{\omega,\lambda}|t|}\,. \tag{3.43}$$

Using Stirling formula, we find, for arbitrary $\mu > 0$ and for $|x| > |t| c_{\omega,\lambda} e^{(\mu/2)+1}$,

$$|H(t,x)| \leq e^{-\mu\left(|x| - \frac{2c_{\omega,\lambda}}{\mu}|t|\right)}\,.$$

Since, by definition $|H(t,x)| \leq 1$ for all $x \in \mathbb{Z}^\nu$ and $t \in \mathbb{R}$, we obtain immediately that

$$|H(t,x)| \leq e^{-\mu\left(|x| - c_{\omega,\lambda} \max\left(\frac{2}{\mu}\,,\, e^{(\mu/2)+1}\right)|t|\right)}$$

for arbitrary $\mu > 0$.

The case $m = 1$ is handled analogously. For the case $m = -1$ we note that

$$H_L^{(-1)}(t,x) = -2 \int_0^t H_L^{(0)}(s,x)ds \tag{3.44}$$

and then use the bound already obtained for the case $m = 0$. $\square$

4 Lieb-Robinson Inequalities for Anharmonic Lattice Systems

In this section we consider perturbations of the harmonic lattice system described by the Hamiltonian H_L^h defined in (3.1). Specifically, for a cube $\Lambda_L = (-L, L]^\nu \subset \mathbb{Z}^\nu$, we consider the anharmonic Hamiltonian

$$\begin{aligned} H_L &= H_L^h + \sum_{x\in\Lambda_L} V(q_x) \\ &= \sum_{x\in\Lambda_L} p_x^2 + \omega^2 q_x^2 + \sum_{x\in\Lambda_L} \sum_{j=1}^{\nu} \lambda_j\, (q_x - q_{x+e_j})^2 + \sum_{x\in\Lambda_L} V(q_x)\,. \end{aligned} \tag{4.1}$$

We denote the dynamics generated by H_L on the algebra $\mathcal{A}_{\Lambda_L}$ by τ_t^L; that is

$$\tau_t^L(A) = e^{itH_L} A\, e^{-itH_L} \qquad \text{for } A \in \mathcal{A}_{\Lambda_L}.$$

The main result of this section will provide estimates in terms of the function

$$F_\mu(r) = \frac{e^{-\mu r}}{(1+r)^{\nu+1}}\,.$$

Since the distance function d is a metric, we clearly have

$$\sum_{z\in\Lambda_L} F_\mu(d(x,z))F_\mu(d(z,y)) \le C_\nu F_\mu(d(x,y)) \tag{4.2}$$

with

$$C_\nu = 2^{\nu+1}\sum_{z\in\Lambda_L}\frac{1}{(1+|z|)^{\nu+1}}. \tag{4.3}$$

Theorem 4.1. *Suppose that $V \in C^1(\mathbb{R})$ is real valued with $V' \in L^1(\mathbb{R})$ such that*

$$\kappa_V = \int \mathrm{d}w\, |\widehat{V'}(w)||w| < \infty\,. \tag{4.4}$$

Then, for every $\mu \ge 1$, and $\epsilon > 0$, there exists a constant C, such that for every pair of finite sets $X, Y \subset \mathbb{Z}^\nu$ and $L > 0$ such that $X, Y \subset \Lambda_L$, we have

$$\Big\| \left[\, \tau_t^L(W(f)), W(g)\,\right] \Big\| \le C\,\|f\|_\infty \|g\|_\infty\, e^{(\mu+\epsilon)v|t|} \sum_{x\in X, y\in Y} F_\mu(d(x,y)) \tag{4.5}$$

for all bounded functions f, g with $\operatorname{supp} f \subset X$ and $\operatorname{supp} g \subset Y$. Here

$$C \;=\; (2 + c_{\omega,\lambda} e^{\frac{(\mu+\epsilon)}{2}} + c_{\omega,\lambda}^{-1})\,\sup_{s\ge 0}\left[(1+s)^{\nu+1}e^{-\epsilon s}\right],$$

and

$$v(\mu+\epsilon) \;=\; v_h(\mu+\epsilon) + \frac{CC_\nu\kappa_V}{\mu+\epsilon}\,,$$

with $v_h(\mu+\epsilon)$ defined in (3.15).

Corollary 4.2. *Analogously to Corollary 3.2, the theorem implies a bound of the form*

$$\Big\| \left[\, \tau_t^L(W(f)), W(g)\,\right] \Big\| \le \tilde{C}\,\|f\|_\infty \|g\|_\infty\, \min(|X|,|Y|)\, e^{-\mu\left(d(X,Y)-(1+\frac{\epsilon}{\mu})v(\mu+\epsilon)|t|\right)}$$

for all $\mu, \epsilon > 0$ and where

$$\tilde{C} = C\sum_{z\in\mathbb{Z}^\nu}\frac{1}{(1+|z|)^{\nu+1}}\,,$$

and $d(X,Y)$ denotes the distance between the supports X and Y.

Proof. We are going to interpolate between the time evolution τ_t^L (generated by the Hamiltonian (4.1)) and the harmonic time evolution $\tau_t^{h;\Lambda_L}$ generated by (3.1); to simplify the notation we will drop all the L dependence in H_L and H_L^h and in the dynamics τ_t^L and $\tau_t^{h;\Lambda_L}$. We start by noting that

$$[\tau_t\left(W(f)\right), W(g)] \;=\; \left[\tau_s\left(\tau_{t-s}^h\left(W(f)\right)\right), W(g)\right]\Big|_{s=t}\,.$$

This leads us to the study of

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\tau_s\Big(\tau^h_{t-s}\Big(W(f)\Big)\Big), W(g)\right] \\
&\qquad = i\left[\tau_s\left(\left[\sum_{z\in\Lambda_L} V(q_z), \tau^h_{t-s}\left(W(f)\right)\right]\right), W(g)\right] \\
&\qquad = i\sum_{z\in\Lambda_L}\left[\,\tau_s\left(\left[V(q_z), W(f_{t-s})\right]\right), W(g)\right]
\end{aligned}
\tag{4.6}
$$

where we used Lemma 3.4 to compute the harmonic evolution of the Weyl operator $W(f)$, and the shorthand notation

$$
f_t = f * \overline{h}^{(L)}_{1,t} + \overline{f} * h^{(L)}_{2,t}
\tag{4.7}
$$

to denote the harmonic evolution of the wave function f. Using (3.9), we easily obtain that

$$
\begin{aligned}
[V(q_z), W(f_{t-s})] &= W(f_{t-s})\left(W^*(f_{t-s})V(q_z)W(f_{t-s}) - V(q_z)\right) \\
&= W(f_{t-s})\left(V(q_z - \mathrm{Im}\ f_{t-s}(z)) - V(q_z)\right) .
\end{aligned}
$$

Inserting the last equation in (4.6) we find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\tau_s\Big(\tau^h_{t-s}\Big(W(f)\Big)\Big), W(g)\right] \\
&\qquad = i\sum_{z\in\Lambda_L}\left[\tau_s\left(W(f_{t-s})\left(V(q_z - \mathrm{Im} f_{t-s}(z)) - V(q_z)\right)\right), W(g)\right] \\
&\qquad = i\sum_{z\in\Lambda_L}\left[\tau_s\left(\tau^h_{t-s}\left(W(f)\right)\right), W(g)\right]\,\tau_s\left(V(q_z - \mathrm{Im} f_{t-s}(z)) - V(q_z)\right) \\
&\qquad\quad + i\sum_{z\in\Lambda_L}\tau_s\left(\tau^h_{t-s}\left(W(f)\right)\right)\,\left[\tau_s\left(V(q_z - \mathrm{Im} f_{t-s}(z)) - V(q_z)\right), W(g)\right] .
\end{aligned}
\tag{4.8}
$$

Next, we define a unitary evolution $\mathcal{U}(s;\tau)$ by

$$
i\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{U}(s;\tau) = \mathcal{L}(s)\mathcal{U}(s;\tau), \quad \text{and } \mathcal{U}(\tau;\tau) = 1
$$

with the time-dependent generator

$$
\mathcal{L}(s) = \sum_{z\in\Lambda_L}\tau_s\left(V(q_z - \mathrm{Im} f_{t-s}(z)) - V(q_z)\right) .
$$

(Here $t \geq 0$ is a fixed parameter). Then, by (4.8), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\tau_s\left(\tau^h_{t-s}\left(W(f)\right)\right), W(g)\right]\mathcal{U}(s;0) \\
&\qquad = i\sum_{z\in\Lambda_L}\tau_s\left(W(f_{t-s})\right)\left[\tau_s\left(V(q_z - \mathrm{Im} f_{t-s}(z)) - V(q_z)\right), W(g)\right]\mathcal{U}(s;0)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\Big[\tau_t\left(W(f)\right), W(g)\Big]\mathcal{U}(t;0)\\
&\qquad = \Big[\tau_t^h\left(W(f)\right), W(g)\Big]\\
&\qquad\quad + i\sum_{z\in\Lambda_L}\int_0^t \mathrm{d}s\;\tau_s\left(W(f_{t-s})\right)\;\left[\tau_s\left(V(q_z-\mathrm{Im}f_{t-s}(z))-V(q_z)\right), W(g)\right]\mathcal{U}(s;0)\,.
\end{aligned}
\tag{4.9}
$$

Next, we expand

$$
\begin{aligned}
\left(V(q_z-\mathrm{Im}f_{t-s}(z))-V(q_z)\right) &= -\,\mathrm{Im}f_{t-s}(z)\int_0^1 \mathrm{d}r\;V'(q_z-r\,\mathrm{Im}f_{t-s}(z))\\
&= -\,\mathrm{Im}f_{t-s}(z)\int_0^1 \mathrm{d}r\int \mathrm{d}w\;\widehat{V'}(w)e^{iw(q_z-r\,\mathrm{Im}f_{t-s}(z))}\,.
\end{aligned}
$$

where the Fourier transform $\widehat{V}'$ is defined as

$$
\widehat{V'}(w)=\int\frac{\mathrm{d}q}{(2\pi)^\nu}\,V'(q)e^{-iq\cdot w}\,.
$$

From (4.9) we obtain

$$
\begin{aligned}
\Big[\tau_t\left(W(f)\right), W(g)\Big] = \Big[\tau_t^h\left(W(f)\right), W(g)\Big]&\mathcal{U}(0;t)\\
-\,i\sum_{z\in\Lambda_L}\int_0^t \mathrm{d}s\;\mathrm{Im}f_{t-s}(z)&\int_0^1\mathrm{d}r\int\mathrm{d}w\,\widehat{V'}(w)\,e^{-iw\,r\,\mathrm{Im}f_{t-s}(z)}\\
&\times\tau_s\left(W(f_{t-s})\right)\left[\tau_s\left(e^{iwq_z}\right), W(g)\right]\mathcal{U}(s;t)\,.
\end{aligned}
$$

Taking the norm, using the unitarity of $\mathcal{U}(s;t))$ and assuming $t\geq 0$ for convenience, we obtain

$$
\begin{aligned}
\Big\|\Big[\tau_t\left(W(f)\right), W(g)\Big]\Big\| &\leq \Big\|\Big[\tau_t^h\left(W(f)\right), W(g)\Big]\Big\|\\
&\quad+\sum_{z\in\Lambda_L}\int_0^t\mathrm{d}s\;|\mathrm{Im}f_{t-s}(z)|\int\mathrm{d}w|\widehat{V'}(w)|\,\Big\|\left[\tau_s\left(e^{iwq_z}\right), W(g)\right]\Big\|\,.
\end{aligned}
\tag{4.10}
$$

For any $\epsilon>0$, it is clear from (3.23) that we have

$$
\begin{aligned}
\Big\|\Big[\tau_t^h\left(W(f)\right), W(g)\Big]\Big\| &\leq\; (2+c_{\omega,\lambda}e^{\frac{(\mu+\epsilon)}{2}}+c_{\omega,\lambda}^{-1})\,\|f\|_\infty\,\|g\|_\infty\,e^{(\mu+\epsilon)v_h(\mu+\epsilon)\,t}\sum_{x\in X,y\in Y}e^{-(\mu+\epsilon)d(x,y)}\\
&\leq\; C\,\|f\|_\infty\,\|g\|_\infty\,e^{\tilde{v}\,t}\sum_{x\in X,y\in Y}F_\mu(d(x,y)),
\end{aligned}
$$

where we have set $\tilde{v}=(\mu+\epsilon)v_h(\mu+\epsilon)$. Similarly, the bound

$$
|\mathrm{Im}f_{t-s}(z)|\;\leq\;C\,\|f\|_\infty\,e^{\tilde{v}(t-s)}\sum_{x\in X}F_\mu(d(z,x)),
\tag{4.11}
$$

follows from an argument as in (3.23), for all $0 \le s \le t$. Plugging these observations into (4.10), we find that

$$\begin{aligned}
&\left\| \left[\tau_t(W(f)), W(g) \right] \right\| \\
&\qquad \le C \, \|f\|_\infty \, \|g\|_\infty \, e^{\tilde{v}\, t} \sum_{x \in X, y \in Y} F_\mu(d(x,y)) \\
&\qquad\quad + C \, \|f\|_\infty \sum_{z \in \Lambda_L} \sum_{x \in X} F_\mu(d(z,x)) \int \mathrm{d}w \, |\widehat{V'}(w)| \int_0^t \mathrm{d}s \, e^{\tilde{v}(t-s)} \left\| \left[\tau_s\left(e^{iwq_z}\right), W(g) \right] \right\| .
\end{aligned}$$

Iterating this inequality m times we obtain

$$\begin{aligned}
&\left\| \left[\tau_t\left(W(f)\right), W(g) \right] \right\| \\
&\quad \le C \, \|f\|_\infty \, \|g\|_\infty \, e^{\tilde{v}\, t} \sum_{x \in X, y \in Y} F_\mu(d(x,y)) \\
&\quad\quad + C \|f\|_\infty \|g\|_\infty e^{\tilde{v} t} \sum_{x \in X, y \in Y} \sum_{n=1}^{m} \frac{(Ct)^n}{n!} \left(\prod_{j=1}^{n} \int dw_j |w_j| |\widehat{V'}(w_j)| \right) \\
&\qquad\qquad \times \sum_{z_1, \dots, z_n \in \Lambda_L} F_\mu(d(x,z_1)) \, F_\mu(d(z_1,z_2)) \dots F_\mu(d(z_n,y)) \\
&\quad\quad + C^{m+1} \, \|f\|_\infty \sum_{x \in X} \left(\prod_{j=1}^{m} \int dw_j |w_j| |\widehat{V'}(w_j)| \right) \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_m} ds_{m+1} \\
&\qquad\qquad \times \sum_{z_1, \dots, z_{m+1} \in \Lambda_L} F_\mu(d(x,z_1)) \, F_\mu(d(z_1, z_2|) \dots F_\mu(d(z_m, z_{m+1})) \\
&\qquad\qquad\qquad \times \int \mathrm{d}w_{m+1} |\widehat{V'}(w_{m+1})| \, e^{\tilde{v}(t - s_{m+1})} \left\| \left[\tau_{s_{m+1}}(e^{iw_{m+1} q_{z_{m+1}}}), W(g) \right] \right\| .
\end{aligned} \tag{4.12}$$

Using (4.2), we find that

$$\sum_{z_1, \dots, z_n \in \Lambda_L} F_\mu(d(x,z_1)) F_\mu(d(z_1,z_2)) \dots F_\mu(d(z_n,y)) \;\le\; C_\nu^n \, F_\mu(d(x,y)).$$

As for the error term in (4.12), we can use the a-priori bound $\|[\tau_{s_{m+1}}(e^{iw_{m+1}q_{z_{m+1}}}), W(g)]\| \le 2$ to obtain

$$\begin{aligned}
&2 \, \|f\|_\infty \, e^{\tilde{v}\, t} \, \|\widehat{V'}\|_1 \, C \, t \frac{(C \, \kappa_V \, C_\nu \, t)^m}{(m+1)!} \sum_{x \in X} \sum_{z_{m+1} \in \Lambda_L} F_\mu(d(x, z_{m+1})) \\
&\qquad\qquad \le \; 2 \, \|f\|_\infty \, e^{\tilde{v}\, t} \, \|\widehat{V'}\|_1 \, C \, t \frac{(C \, \kappa_V \, C_\nu \, t)^m}{(m+1)!} |X| \sum_{z \in \mathbb{Z}^\nu} F_\mu(|z|).
\end{aligned}$$

From (4.12), we now conclude that

$$
\begin{aligned}
\Big\| \Big[\tau_t \left(W(f) \right), W(g) \Big] \Big\| &\leq C \, \|f\|_\infty \|g\|_\infty \, e^{\tilde{v} t} \sum_{x \in X, y \in Y} F_\mu(d(x,y)) \sum_{n \geq 0} \frac{(C \, \kappa_V \, C_\nu \, t)^n}{n!} \\
&\quad + 2 \, \|f\|_\infty \, e^{\tilde{v}\, t} \, \|\widehat{V'}\|_1 \, C \, t \frac{(C \, \kappa_V \, C_\nu \, t)^m}{(m+1)!} |X| \sum_{z \in \mathbb{Z}^\nu} F_\mu(|z|) \\
&\leq C \, \|f\|_\infty \|g\|_\infty \, e^{(\tilde{v} + C \, \kappa_V \, C_\nu) t} \sum_{x \in X, y \in Y} F_\mu(d(x,y)) \\
&\quad + 2 \, \|f\|_\infty \, e^{\tilde{v}\, t} \, \|\widehat{V'}\|_1 \, C \, t \frac{(C \, \kappa_V \, C_\nu \, t)^m}{(m+1)!} |X| \sum_{z \in \mathbb{Z}^\nu} F_\mu(|z|) \, .
\end{aligned}
$$

Since this is true for every $m \geq 0$, and since the last term converges to zero as $m \to \infty$, the theorem follows. $\square$

Remark 4.3. *Exactly the same proof yields the Lieb-Robinson bounds (4.5) for the Hamiltonian*

$$
\widehat{H}_L = \sum_{x \in \Lambda_L} p_x^2 + \omega^2 \, q_x^2 + \sum_{x \in \Lambda_L} \sum_{j=1}^{\nu} \lambda_j \, (q_x - q_{x+e_j})^2 + \sum_{x \in \Lambda_L} V(p_x) \, .
$$

Moreover, one can see from the proof that the on-site nature of the anharmonic perturbation does not play an important role here. For example the same technique can be used to establish Lieb-Robinson bounds for the dynamics generated by the Hamiltonian

$$
\widetilde{H}_L = \sum_{x \in \Lambda_L} p_x^2 + \omega^2 \, q_x^2 + \sum_{x \in \Lambda_L} \sum_{j=1}^{\nu} \lambda_j \, (q_x - q_{x+e_j})^2 + \sum_{x \in \Lambda_L} \sum_{j=1}^{\nu} \big(V_1(q_x - q_{x+e_j}) + V_2(p_x - p_{x+e_j}) \big)
$$

if both V_1 and V_2 satisfy the assumption (4.4).

5 Discussion

5.1 Other Observables

Theorems 3.1 and 4.1 give a Lieb-Robinson bound for Weyl operators of the form

$$
\|[\tau_t(W(f)), W(g)]\| \leq C \|f\|_\infty \|g\|_\infty e^{-\mu(d(X,Y) - v|t|)} \tag{5.1}
$$

for f and g supported on finite subsets X and Y of the lattice, where τ_t is the dynamics of a harmonic or anharmonic lattice system that satisfies the conditions of these theorems. From (5.1) one can of course immediately obtain a bound for observables A and B that are finite linear combinations of Weyl operators by a simple application of the triangle inequality. Two other classes of observables for which we can obtain useful bounds are worth mentioning.

Note that for every $f : X \to \mathbb{C}$, $W(f) = e^{ib(f)}$, with a self-adjoint operator $b(f)$ acting on $\mathcal{H}_X$ (3.6), such that $b(sf) = sb(f)$ for every $s \in \mathbb{R}$. Let $\hat{A}, \hat{B} \in L^1(\mathbb{R})$ be two functions such

that $s\hat{A}(s)$ and $s\hat{B}(s)$ are also in $L^1(\mathbb{R})$. Then, it is straightforward to derive a Lieb-Robinson bound for the observables $A(b(f))$ and $B(b(g))$ defined by

$$A(b(f)) = \int ds \hat{A}(s) W(sf), \quad B(b(g)) = \int ds \hat{B}(s) W(sg)\,. \tag{5.2}$$

The result is

$$\|[\tau_t(A(b(f))), B(b(g))]\| \leq C\|f\|_\infty \|g\|_\infty e^{-\mu(d(X,Y)-v|t|)} \int ds|s\hat{A}(s)| \int ds|s\hat{B}(s)| \tag{5.3}$$

By taking derivatives, we can also obtain a Lieb-Robinson bound for the unbounded observables $b(f)$ and $b(g)$ (e.g., q_x and p_x). Because $b(f)$ and $b(g)$ are unbounded we apply the Lieb-Robinson bound first on a common dense domain of analytic vectors (see [5, Lemma 5.2.12]), and find that the commutator $[\tau_t(b(f)), b(g)]$ has a bounded extension with the following norm bound

$$\|[\tau_t(b(f)), b(g)]\| \leq C\|f\|_\infty \|g\|_\infty e^{-\mu(d(X,Y)-v|t|)}\,. \tag{5.4}$$

5.2 Exponential Clustering Theorem

For a large class of quantum spin systems it was recently proven that a non-vanishing spectral gap implies exponential decay of spatial correlations in the ground state [18, 14, 21]. Such a result is often referred to as the Exponential Clustering Theorem. The locality property of the dynamics provided by a Lieb-Robinson bound is one of the main ingredients in the proof of this result. In the harmonic case, the clustering properties of the exact ground state can be explicitly analyzed [9, 23], and indeed one finds exponential decay whenever there is a non-vanishing gap. For the harmonic systems considered here, the gap is non-vanishing iff $\omega > 0$. The results of this paper can be used to prove an exponential clustering theorem for the class of anharmonic lattice systems we consider here. In fact, following the method of [21] (see also [18, 14]), the only additional estimate needed is the following short-time bound.

Lemma 5.1. *Let H_L be the Hamiltonian acting on $\Lambda_L = (-L, L]^\nu \subset \mathbb{Z}^\nu$ defined in (4.1), and τ_t^L the time-evolution generated by H_L. Let $f, g : \Lambda_L \to \mathbb{R}$ with $\operatorname{supp} f \subset X$, $\operatorname{supp} g \subset Y$, and $X \cap Y = \emptyset$. Then there exists a constant $C = C(\lambda, \omega, \kappa_V) < \infty$ such that*

$$\|[\tau_t(W(f)), W(g)]\| \leq C\,|t|\, \min(|X|, |Y|)\, \||f\|_\infty \|g\|_\infty \tag{5.5}$$

for all $|t| < t_0(\lambda, \omega, \kappa_V)$.

Proof. Let $H_L^{(m)}(t,x)$, for $m = 0, \pm 1$, be the Fourier sums defined in (3.37). From (3.43), we obtain that, for arbitrary $\mu > 0$,

$$|H^{(0)}(t,x)| \leq (2c_{\omega,\lambda}|t|) \frac{(2c_{\omega,\lambda}|t|)^{2|x|-1}}{(2|x|!)} \leq c_{\omega,\lambda}\,|t|\, e^{(\mu/2)+1}\, e^{-\mu(|x|-\frac{2c_{\omega,\lambda}}{\mu}|t|)}$$

for all $|x| \geq 1$ and $|t| < e^{-(\mu/2)-1} c_{\lambda,\omega}^{-1}$. Since similar estimates hold for $H^{(1)}$ and $H^{(-1)}$ as well, we find, analogously to (3.23), that, if τ_t^h denotes the harmonic time-evolution generated by the Hamiltonian (3.1),

$$
\begin{aligned}
\left\| \left[\tau_t^h \left(W(f) \right), W(g) \right] \right\| &\leq C\, |t| \, \|f\|_\infty \|g\|_\infty \sum_{x \in X, y \in Y} e^{-\mu \left(d(x,y) - \frac{2c_{\omega,\lambda}}{\mu} |t| \right)} \\
&\leq C\, |t| \, \|f\|_\infty \|g\|_\infty \min(|X|, |Y|)
\end{aligned}
\tag{5.6}
$$

for all $|t| < e^{-(\mu/2)-1} c_{\omega,\lambda}^{-1}$ (using the assumption that $X \cap Y = \emptyset$), and for a constant C depending only on λ and ω.

Next we consider the anharmonic time evolution $\tau_t \equiv \tau_t^L$. From (4.10), it follows that

$$
\begin{aligned}
\left\| \left[\tau_t \left(W(f) \right), W(g) \right] \right\| \leq & \left\| \left[\tau_t^h \left(W(f) \right), W(g) \right] \right\| \\
& + \sum_{z \in \Lambda_L} \int_0^t ds \, |\mathrm{Im} f_{t-s}(z)| \int dw |\widehat{V'}(w)| \left\| \left[\tau_s \left(e^{iwq_z} \right), W(g) \right] \right\| .
\end{aligned}
\tag{5.7}
$$

Applying (5.6) to bound the first term, (4.11) and Corollary 4.2 to bound the second term, we find

$$
\left\| \left[\tau_t \left(W(f) \right), W(g) \right] \right\| \leq C\, |t| \, \|f\|_\infty \|g\|_\infty \min(|X|, |Y|) \tag{5.8}
$$

for a constant C depending only on λ, ω and on the constant κ_V defined in (4.4), and for all $|t|$ sufficiently small (depending on λ, ω, and κ_V). $\square$

As a consequence of these considerations one obtains the following theorem.

Theorem 5.2. *Let H be the Hamiltonian of a harmonic or anharmonic lattice model satisfying the conditions of Theorem 3.1 or 4.1, and suppose H has a unique ground state Ω and a spectral gap γ above the ground state. Denote by $\langle \cdot \rangle$ the expectation in the state Ω. Then, for any functions f and g with supports X and Y in the lattice we have the following estimate:*

$$
|\langle W(f) W(g) \rangle - \langle W(f) \rangle \langle W(g) \rangle| \leq C \|f\|_\infty \, \|g\|_\infty \, \|\widehat{V'}\|_1 \min(|X|, |Y|) e^{-d(X,Y)/\xi} \tag{5.9}
$$

where $\mu \geq 1$ and $\epsilon > 0$ are as in Theorem 4.1 and ξ can be taken to be

$$
\xi = \frac{2(\mu + \epsilon) v(\mu + \epsilon) + \gamma}{\mu \gamma} \tag{5.10}
$$

and where, if we assume $d(X,Y) \geq \xi$, C is a constant depending only on the lattice.

It is straightforward to see that the same bound holds for infinite systems if the corresponding GNS Hamiltonian has a unique ground state and a spectral gap above it, and the infinite system is the thermodynamic limit of finite systems that satisfy the conditions of Theorem 3.1 or 4.1.

Acknowledgements

This article is based on work supported in part by the U.S. National Science Foundation under Grant # DMS-0605342 (B.N. and R.S.). H.R. received support from NSF Vigre grant #DMS-0135345. B.S. is on leave from University of Cambridge; his research is supported by a Sofja Kovalevskaja Award of the Humboldt Foundation.

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